Table 2 Errors in natural frequencies of simply supported beam models

Mode	N	Lumped mass f_n [a]	Lumped mass ϵ [b]	Lumped mass ϵ_t, Eq. (8)	Consistent mass f_n [a]	Consistent mass ϵ [b]	Consistent mass ϵ_t, Eq. (9)
1	8	0.9999828	−0.0000172	−0.0000165	1.000016	0.000016	0.000016
2	4	3.998780	−0.000305	−0.000264	4.001039	0.000260	0.000264
3	8/3	8.983501	−0.001833	−0.001388	9.011579	0.001286	0.001388
4	2	15.88395	−0.007259	−0.004228	16.06315	0.003947	0.004228
5	8/5	24.42284	−0.023086	−0.010322	25.23177	0.009271	0.010322
6	4/3	33.72486	−0.063198	−0.021403	36.65781	0.018272	0.021403
7	8/7	41.65990	−0.14980	−0.039652	50.52612	0.031145	0.039652

[a] Frequency calculated with NASTRAN. [b] Error in f_n.

estimates, Eqs. (8) and (9) and those obtained from NASTRAN calculations, are compared in Table 2. The large differences between the error estimates generated by NASTRAN and those computed with Eq. (8), especially for the higher modes, are due to terms $O(kL)^8$ neglected in the kinetic energy error estimation. These terms become significant as $(\pi/N)^4$ approaches unity, that is, as the number of elements per half wavelength decreases.

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Pressure Dependence of Burning Rate of Composite Solid Propellants

R. P. Rastogi,* H. J. Singh,† Desh Deepak,‡ and A. K. Srivastava§

University of Gorakhpur, Gorakhpur, India

Nomenclature

a, b = constants of Eq. (15)
C_f = concentration of species f
C_{fe} = equilibrium concentration of species f at the end of chemical reaction zone
C_{fs} = concentration of species f near the propellant surface
C_s = specific heat of solid (average of fuel and oxidizer)
k_0 = rate constant at zero atmosphere
k_1 = rate constant of the first-order reaction
L = length of the chemical reaction zone
$\dot{m}$ = mass buring rate
n_f = number of moles of fuel
P = chamber pressure
Q_h = net heat release (positive) for gasification of the propellant
$\dot{r}$ = buring rate
R = gas constant
R_f = chemical reaction rate per unit volume
T_0 = ambient temperature
T_f = flame temperature at the end of chemical reaction zone
T_s = surface temperature of the burning propellant (average)
U = volume rate of flow
V = total volume of the reactor
ΔV^* = volume of activation, i.e., difference in the volumes of activate complex and the reactants
λ_{gs} = average thermal conductivity of flame gas at the surface
ρ_g = density of the gas
ρ_p = propellant density

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*Professor, Dept. of Chemistry. Associate Fellow AIAA.

†Lecturer, Dept. of Chemistry.

‡Presently at Hindustan Lever Ltd., Research Centre, Andheri (East), Bombay.

§Junior Research Fellow, Dept. of Chemistry.

Introduction

VARIOUS equations relating the burning rate of composite propellants and pressure have been proposed from time to time.[1] Attempts have been made by several workers[2-6] to develop theories of combustion of solid propellants. Rastogi et al.[7] have recently reviewed the various theories of combustion of propellants. Out of these the theory based on the granular diffusion flame model is most convenient for deducing expressions for the pressure dependence of burning rate. A wide range of experimental data is available and none of the proposed equations fits the entire range. Hence, it is desirable to re-examine the question of pressure dependence of burning rate.

We shall base our discussion on the concept of heat balance at the propellant surface, that is,

$$\dot{m}[C_s(T_s - T_0) - Q_h] = \lambda_{gs}\frac{(T_f - T_s)}{L} \tag{1}$$

In combustion theories L has been introduced in a rather phenomenological way and has been related to chemical reaction rate in a dubious manner. The pressure dependence of $\dot{m}$ appears to follow as a consequence of pressure dependence of density.[6] In the following theory, we have considered the reaction zone as a nonisothermal reactor and the pressure dependence of the reaction rate constant. $\dot{m}$ comes out to be pressure dependent on account of the latter. The theoretical results have been compared with the burning rate data at various pressures for different propellants having difference particle sizes of the oxidizer. The available ex-

perimental data are found to be in agreement with the theory developed.

Theory

In the steady-state burning of a solid propellant, continuous inflow of reactant vapors into the reaction zone is maintained. The energy needed for the vaporization of the propellant is obtained from the heat transferred from the flame zone to the propellant surface. This energy feedback is mainly by conduction and radiation. The latter is neglected in this study since it makes the analysis simpler. Further, it is assumed that total heat is released in the reaction zone where oxidizer and fuel species react together exothermically and also that no chemical reaction occurs in the solid phase.

The chemical reaction zone over the burning propellant surface may be considered to be a tubular flow reactor.[8] In this reactor fuel vapor and gaseous oxidizing species enter continuously from the propellant surface, and combustion products are formed which leave the other end of the reactor.

Let us assume that the concentration of fuel vapor entering the volume element dV is C_f while the concentration of fuel vapor leaving the element is $C_f - dC_f$. The net rate of change in the amount of fuel within the volume element would be equal to (dn_f/dt), which will be the sum of two terms: 1) due to chemical reaction within the element, and 2) due to fuel velocity. In case of an isothermal tubular reactor,[8]

$$dn_f/dt = R_f dV - U dC_f \tag{2}$$

In the steady-state (dn_f/dt) would be equal to zero. Thus, Eq. (2) takes the form

$$R_f dV - U dC_f = 0 \tag{3}$$

Assuming that reaction is first order with respect to f, $R_f = -k_1 C_f$. Thus, Eq. (3) becomes

$$-k_1 C_f dV - U dC_f = 0 \tag{4}$$

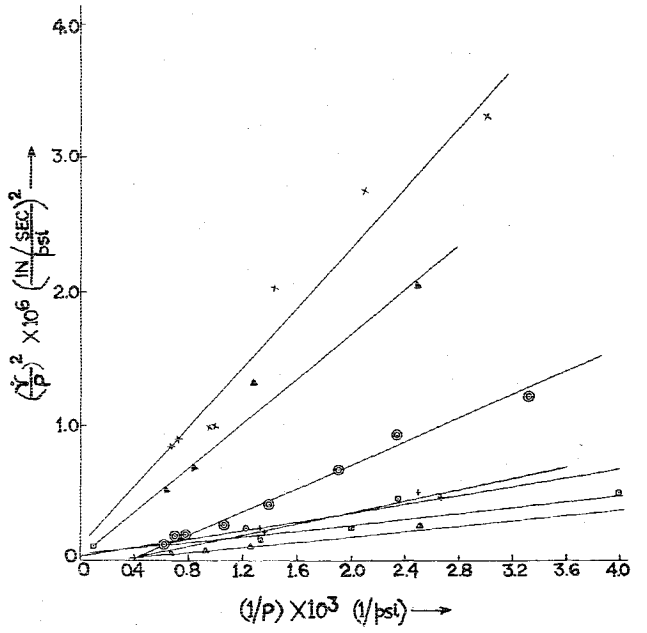

Fig. 1 Least-square plots of $(\dot{r}/P)^2$ vs $(1/P)$ for: ▲ 35% LP3/GMF + (30% 9μ + 70% 265μ) AP; ◎ 20% PBAA/EPON + (30% 5μ + 70% 45μ) AP; ⊙ 25% P13 + (16μ) AP; ◬ 32.5% EPON/TETA + (10μ) AP; X 60% NC/TEGDN + (5μ)AP; + 25% P13 + (50% 70μ + 50% 78μ)AP; ⊡ 14% PBAN + AP. (The abbreviations for fuels have been expressed in Ref. 6.)

Since reaction sites are distributed all over the space available in the reactor, there would exist a temperature gradient. Thus, the system would correspond to a nonisothermal flow reactor. In such a case Eq. (4) can be modified as

$$-k_1 C_f dV\, dT - U_T dC_f dT = 0 \tag{5}$$

U_T, the volume flow rate of the gaseous mixture, will depend on temperature, and can be written as

$$U_T = \frac{\dot{m}}{\rho_g} = \frac{\dot{m} TR}{P} \tag{6}$$

since, $P/\rho_g = RT$ for ideal gases.

With the help of Eq. (6) we can rewrite Eq. (5) as

$$-k_1 C_f dV\, dT - \frac{\dot{m} TR}{P} dC_f dT = 0 \tag{7}$$

The rate constant k_1 would depend on pressure, and can be written as[9]

$$k_1 = k_0 \exp(-P\Delta V^*/RT) \tag{8}$$

Combining Eqs. (7) and (8), we find

$$-k_0 \exp\left(\frac{-P\Delta V^*}{RT}\right) dV\, dT - \frac{\dot{m} TR}{P} \frac{dC_f}{C_f} dT = 0 \tag{9}$$

Expanding the exponential term of Eq. (9) and retaining the first term of expansion, we get

$$-k_0 \left(1 - \frac{P\Delta V^*}{RT}\right) dV\, dT = \frac{\dot{m} TR}{P} \frac{dC_f}{C_f} dT \tag{10}$$

By integrating Eq. (10) under the proper limits, one would obtain

$$\int_0^V \int_{T_s}^{T_f} -k_0 \left(1 - \frac{P\Delta V^*}{RT}\right) dV\, dT = \int_{C_{fs}}^{C_{fe}} \int_{T_s}^{T_f} \frac{\dot{m} TR}{P} \frac{dC_f}{C_f} dT$$

Or,

$$k_0 V \left[(T_f - T_s) - \frac{P\Delta V^*}{R} \ln \frac{T_f}{T_s}\right] = \frac{\dot{m} R}{2P} (T_f^2 - T_s^2) \ln \frac{C_{fs}}{C_{fe}} \tag{11}$$

Assuming the area of the cross section of the reactor to be unity and L the length of the reaction zone, the volume V should be equal to L. Thus Eq. (11) takes the form

$$L = \frac{R\dot{m}(T_f^2 - T_s^2) \ln(C_{fs}/C_{fe})}{2k_0 P[(T_f - T_s) - (P\Delta V^*/R) \ln(T_f/T_s)]} \tag{12}$$

Using the heat balance at the propellant surface as given by Eq. (1) and combining it with Eq. (12), we get an expression for $\dot{m}$, as,

$$\dot{m}^2 = \frac{2\lambda_{gs}(T_f - T_s) k_0 P[(T_f - T_s) - (P\Delta V^*/R) \ln(T_f/T_s)]}{R[C_s(T_s - T_0) - Q_h](T_f^2 - T_s^2) \ln(C_{fs}/C_{fe})} \tag{13}$$

Since $\dot{m} = \rho_p \dot{r}$, the pressure dependence of burning rate emerges from Eq. (13) in the following form,

$$\dot{r}^2 = \left\{\frac{2\lambda_{gs}(T_f - T_s)^2 k_0}{R\rho_p^2 [C_s(T_s - T_0) - Q_h](T_f^2 - T_s^2) \ln(C_{fs}/fe)}\right\} P$$

$$- \left\{\frac{2\lambda_{gs}(T_f - T_s) k_0 V^* \ln(T_f/T_s)}{R^2 \rho_p^2 [C_s(T_s - T_0) - Q_h](T_f^2 - T_s^2) \ln(C_{fs}/C_{fe})}\right\} P^2$$

Or,

$$\dot{r}^2 = aP - bP^2 \tag{14}$$

where

$$a = \frac{2\lambda_{gs}(T_f - T_s)^2 k_0}{\rho_p^2 R[C_s(T_s - T_0) - Q_h](T_f^2 - T_s^2)\ell n(C_{fs}/C_{fe})}$$

and

$$b = \frac{2\lambda_{gs}(T_f - T_s)k_0 V^* \ell n(T_f/T_s)}{\rho_p^2 R^2[C_s(T_s - T_0) - Q_h](T_f^2 - T_s^2)\ell n(C_{fs}/C_{fe})}$$

Eq. (14) can be rearranged to give

$$(\dot{r}^2/P)^2 = (a/P) - b \tag{15}$$

Discussion

In order to test the validity of Eq. (15), we have examined the available data on pressure dependence of burning rate of composite solid propellants. Values of burning rates have been obtained from Fig. 2, of Ref. 6. The values of $(\dot{r}/P)^2$ have been plotted against $1/P$ in Fig. 1 for propellants having binders of different types. The propellants chosen are highly AP loaded where AP particles sizes are of intermediate size (10-70 μ). The plots are straight lines showing that Eq. (15) is valid. Equation (15) has also been verified for highly loaded bimodal AP propellant and the results will be published elsewhere.

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Linearized Solution of Conducting-Radiating Fins

Robert D. Karam* and Robert J. Eby†
Fairchild Space and Electronics Company, Germantown, Md.

Nomenclature

A_c	= cross-sectional area
A_i	= area receiving heat flux
A_r	= radiating area
k	= thermal conductivity
L	= fin length
Q	= total heat rejected
$q(x)$	= heat flux input
T	= absolute temperature
T_0	= temperature at $x=0$
T_m	= mean temperature
U	= constant heat flux input
x	= distance
α,β,γ	= dimensionless parameters
ϵ	= emittance
ξ	= dimensionless distance
σ	= Stefan-Boltzmann constant
τ	= dimensionless temperature
τ_m	= dimensionless mean temperature

Introduction

HEAT transfer calculations for conducting-radiating fins can be considerably simplified by replacing the temperature to the fourth power in the radiation term with a linear expansion about a parameter T_m known as a "mean temperature" (see, for example, Refs. 1-3). The solution of the linearized steady-state energy equation is given in the following discussion, and a method is described by which T_m is optimized as a function of fin properties in order to minimize the errors introduced by the process of linearization. The technique is applied to the problem of a fin under a constant flux environment, and results are presented in graphical form suitable for engineering calculations.

Analysis

Linearization of Energy Equation

Consider a one-dimensional fin of constant properties and a uniform cross section (Fig. 1). Without loss of generality, radiation may be assumed to a space environment ($0R$) and the boundary conditions may be taken as

$$T(0) = T_0 \text{ and } dT/dx)_{x=L} = 0$$

The steady-state energy equation is

$$\frac{d^2T}{dx^2} + \frac{A_i q(x)}{kA_c L} - \frac{\epsilon A_r \sigma T^4}{kA_c L} = 0 \tag{1}$$

T^4 in Eq. (1) is approximated by the first two terms of a Taylor expansion about some T_m which is to be determined; that is,

$$T^4 \approx T_m^4 + 4T_m^3(T - T_m) \tag{2}$$

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*Principal Engineer, Thermal Control.
†Manager, Thermal Control.